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REVIEWS

Continuum Mechanics. Concise Theory and Problems. By P. CHADWICK. George Allen & Unwin, 1976. 174 pp. £5.25.

When reviewing this book I was reminded of the Browning poem 'Childe Roland to the Dark Tower Came'. In the poem the questing hero undergoes many arduous and frightening trials in a dismal, barren landscape, but the squat brown tower he finds at the end of the trail is shockingly prosaic. Metaphorically equating the trials of the knight to the large amount of tedious algebra in the book, I think the solution of the Poiseuille flow on page 150 of this book corresponds fairly well to a squat brown tower. To show the main reason for the dullness of the book I quote from the Preface:

> The book is intended primarily for use in conjunction with a lecture course, or equivalent form of teaching. The treatment of the principles of continuum mechanics, while reasonably complete mathematically, is, by design, concise; and to a considerable extent physical considerations, motivating arguments and detailed interpretations have been omitted. In matters of this kind the individual teacher will wish to exercise his own points of view: if the time at his disposal is increased by the availability to his students of this plain account of the theoretical groundwork, I shall be well satisfied.

The presentation takes one through familiar chapters on vector and tensor theory, basic kinematics, balance laws, field equations and jump conditions, with a final chapter on constitutive equations. As the Preface makes clear, it is not claimed that the book conveys understanding of the physical aspects of continuum mechanics without an expository lecture course. Inspection of the book shows that this is undoubtedly true. I have not found any errors or absurdities in the text but, writing on behalf of the readers of the *JFM*, I am led to wonder why we need *another* book of this type. There are already many which are reasonably concise and mathematically equivalent (e.g. D. C. Leigh's *Nonlinear Continuum Mechanics*, McGraw Hill, 1968), and which contain much more in the way of physical explanation, connexions with experiment, and applications. Despite claims of mathematical completeness, certain essential and interesting points are not derived in the present text; for example, on pages 142–143, for the reduction of a tensor function to the useful Reiner–Rivlin form we are referred to one of the few references quoted and no explanation is given. Of course, this is not much of a criticism, since many will be happy to skip such steps anyway. What is more disturbing is the conflict with my belief that the subject of continuum mechanics needs people who are interested in complex physical phenomena and their explanation via simple mathematics; here the converse is presented and I doubt if the present volume will help to recruit bright young people to the subject. This I am unfortunately not able to suggest a use for this book, apart from the original purpose of complementing the author's lectures.

R. I. TANNER

The Finite Element Method for Engineers. By KENNETH H. HUEBNER. Wiley, 1975. 500 pp. $21.95.

Numerical methods for the solution of problems in fluid mechanics have been in use for over 50 years. The oldest and most widely employed method is that based on the use of finite differences (FDM). The father of numerical fluid dynamics is undoubtedly Lewis Fry Richardson, who performed a manual numerical weather forecast using finite-difference methods during his idle moments as an ambulance driver in World War I.

More recently, methods known as spectral or pseudo-spectral methods have been developed. For certain classes of problem, these proved to be superior to finite-difference methods, in that they yielded comparable accuracy with a reduced computational effort, or alternatively considerably better accuracy for the same computational effort. However, restrictions on the geometries that these methods can handle, and the inherent dangers arising from an injudicious selection of base function, may have prevented these methods from being as widely employed as their power would appear to warrant.

In the last half dozen years, there has been a rapidly growing interest in a third method, the finite-element method, for the solution of fluid mechanics problems. Originating some fifteen or so years ago in the aircraft industry as a tool for the analysis of complex aircraft structures, the finite-element method (FEM) is now recognized as being applicable to almost all continuum or field problems.

Both the FDM and the FEM have as their objective the reduction of the governing partial differential equation or equations of some problem to a system of algebraic equations – normally a linear system – in which the unknowns are approximations to the solution variables at a finite set of mesh points or nodes which are more or less uniformly distributed throughout the solution region. The simplest description of the difference between the two methods is this: whereas the FDM uses a *pointwise* approximation to the governing equations, the FEM uses a *piecewise* approximation over a small section of the solution region which is known as an element. The basic premise of the FEM is that a solution region can be analytically modelled or approximated by replacing it with an assemblage of discrete elements. Since these elements can be put together in a variety of ways, they can be used to represent exceedingly complex shapes. The book under review is a comprehensive and generally lucid exposition of the techniques available for performing this modelling process.

Huebner has aimed at producing a 'starting point' text that presents the FEM at an easy-to-understand, introductory level. The book contains a good mixture of theoretical and computational topics, and a very welcome 'practical' chapter which gives a detailed description of a computer program employing the FEM to solve a heat conduction problem.

The book opens with a short chapter entitled "Meet the Finite Element Method", which contains 115 references including 17 conferences and short courses and 7 books which are devoted largely or entirely to the FEM. (The book as a whole is liberally endowed with references.)

Chapter 2, on "The Direct Approach: A Physical Interpretation", opens with a rather trivial discussion of some simple linear elements and their properties, continues with a discussion of the triangular element – the most widely used element in two-dimensional problems – and returns to a one-dimensional problem for a description of the method for the assembly of the element equation into the system of equations for the solution region as a whole. From a didactic point of view, I found this chapter rather confused and disjointed, and I suspect that a lecturer using this book as his text may have to guide his students around the chapter, rather than through it.

In chapter 3, methods for deriving element equations from variational principles are described. Attention is focused on the Rayleigh–Ritz method, which is a forerunner of the FEM, and which possesses many features in common with it. The variational approach is illustrated by the solution of the Reynolds–Poisson equation for the pressure distribution in a fluid between two flat parallel plates which are approaching each other at a uniform speed. A clear, step-by-step description of the solution process is given.

Of course, variational methods can only be used if a variational principle exists for the problem under consideration. Huebner suggests three methods for determining whether this is the case: use mathematical manipulation, use Frechet derivatives (to test for the existence of a variational principle) or consult mathematics textbooks on variational methods and try to find the problem in question and its corresponding variational principle! Unfortunately, most of theinteresting problems in fluid mechanics do not possess a classical variational principle, and so none of these methods is likely to be successful.

Chapter 4 shows how finite-element equations can be derived directly from the governing differential equations of the problem without reliance on any classical, quasi-variational, or restricted variational 'principles'. Two procedures are given to accomplish this; one employs the method of weighted residuals, while the other stems from a global energy balance concept. Among the weighted residual techniques, Huebner states that the error distribution principle most often used to derive finite-element equations is Galerkin's method. This may be so, but some important work has also been done using the least-squares criterion, which might have been mentioned. The development of finite-element equations from global energy balances is described briefly, and in very general terms.

Chapter 5 is devoted to a discussion of many of the various possible two-dimensional and three-dimensional finite elements, and to some of the interpolation functions which can be used to describe the behaviour of the solution variable within elements. Many new concepts are introduced in this chapter, and it is here that we get down to the 'real' finite-element method, in contrast with the more elementary treatment given in the earlier chapters.

The remaining five chapters constituting slightly over half the 500 pages of the book, present applications of the FEM to a range of problems. Chapter 6 discusses elasticity problems, and chapter 7 discusses general field problems. Of more interest to readers of this journal are chapter 8, on the lubrication problem, and chapter 9, on fluid mechanics problems.

Chapter 8 opens with the derivation of the Reynolds equation for incompressible hydrodynamic lubrication; the energy equation, needed to include the consideration of the variation of fluid properties with temperature, is presented. When the lubricant viscosity is independent of the co-ordinate across the film thickness, the Reynolds equation has a variational principle. The development of the finite-element equations from this principle is then presented in some detail. Consideration is given to the deformation of the bearing surfaces under the influence of hydrodynamic pressure. Thermohydrodynamic lubrication is analysed using Galerkin's method, and gas lubrication is also briefly discussed. The chapter concludes by giving some sample solutions, although Huebner points out that, for proprietary reasons, some of the solutions which have undoubtedly been found in industry do not appear in the literature. The examples presented include incompressible isothermal solutions, solutions with compliant bearing surfaces, incompressible thermohydrodynamic solutions and compressible solutions.

In chapter 9, the FEM is applied to more general fluid mechanics problems. Heubner issues the caution that the FEM can help to alleviate difficulties associated with complex geometries, multiply connected domains, and complex boundary conditions, but "should not be expected to triumph in every case where finite difference methods have failed". He points out that, with either method, the accurate numerical solution of most viscous flow problems requires vast amounts of computer time and data storage.

Clearly, we are treading here on the edge of the domain explored by the FEM. "None of the example problems presented really illustrates the full potential of the FEM in fluid mechanics because all of these problems have been solved by using finite difference techniques. Instead, the problems serve as test cases that demonstrate feasibility." In fact, there are now solutions available in the literature to problems which have not been solved by the FDM. Some may be found, for example, in *Finite Elements in Fluids* (edited by R. H. Gallagher, J. T. Oden, C. Taylor & O. C. Zienkiewicz, Wiley, 1975; reviewed in *J. Fluid Mech.* vol. 78, 1976, p. 639), the two volumes of which contain 27 of the papers presented at a symposium held at Swansea in 1974 on the FEM in flow problems. Others have appeared in recent issues of *Int. J. Num. Methods Engng*, *Computers & Fluids*, *A.I.A.A. J.* and even *JFM*.

The chapter opens with a discussion of some inviscid incompressible flows: potential flow is in the class of problems for which a variational principle exists. Problems of flow around one or more bodies, the Kutta condition and some free-surface flows are discussed. Some finite-element analyses of inviscid compressible flows have been made; they have been restricted mainly to subsonic flows, since FEM's to handle the hyperbolic equation and shock fronts are not yet well developed. Details of the variational formulation are provided. Examples include the flow in two-dimensional and three-dimensional nozzles, and over two-dimensional and three-dimensional aerofoils. The latter problem, using 5000 hexahedral elements, was solved in 15 min of central processor time on an IBM 370/165 computer.

The next section of the chapter discusses low Reynolds number viscous flow in which the inertia forces are neglected. The stream function formulation of this problem (i.e. the biharmonic equation), which in theory is appealing, is difficult to use in practice since it places restrictions on the elements which are difficult to satisfy. The primitive variable formulation (u, v, p) is preferable, and is discussed in some detail. Developing and fully developed laminar flow in a channel have been studied using the theory presented.

As with FDM, the inclusion of the inertia terms adds to the difficulties of solution. No classical variational principle exists for the full Navier–Stokes equations for steady incompressible viscous flow, but a pseudo-variational statement can be found under certain conditions. Alternatively, the equation can be linearized by the use of a combination of 'old' and 'new' time step quantities, and a variational statement can then be obtained. Examples of flow over a cylinder and through a partially restricted passage are discussed. Another example describes the use of the FEM to solve the primitive variable equations for the flow past a square cavity.

A brief outline is given of the procedure for treating compressible viscous flows. The chapter concludes with 51 cited references and 42 additional references.

The final chapter is entitled "A Sample Computer Code and Other Practical Considerations". It is addressed to "readers who have never programmed a digital computer to solve a continuum problem by the finite element method". The program was written to solve the steady-state heat conduction problem for two-dimensional or axisymmetric three-dimensional bodies of arbitrary shape and inhomogeneous composition. The necessary theory, which was given in an earlier chapter, is summarized and then the computer program is described in considerable detail. The program listing itself is given, following which are several examples of its use. Mention is made of automatic mesh generation (which can alleviate one of the more tiresome aspects of using the FEM), a discussion is given of numerical integration formulae, some references are provided to methods for the solution of large systems of equations and a list of publicly available finite-element computer programs is given.

The book concludes with four appendices: matrices (9 pages), variational calculus (10 pages), basic equations from linear elasticity theory (21 pages) and basic equations from fluid mechanics (18 pages). The book is well produced, with few misprints, although some of the illustrations are rather poor (unfortunately, in the fluid mechanics chapter), and one – figure 8.2 – looks like the creation of an advertising manager.

It seems that there is one outstanding advantage and one outstanding disadvantage of the FEM over the FDM. First the bad news: the FEM demands of the practitioner a somewhat greater level of mathematical knowledge (perhaps not such a bad feature!) and a considerably greater effort in preparing his problem for solution on a computer than does the FDM. The rigorous analysis of FDM's can be a highly challenging mathematical task; indeed, there are still unanswered questions about methods in common use. However, their *implementation* is usually fairly straightforward, a claim which can not be made for FEM's.

Now for the good news: the FEM is much more suited than the FDM to solving problems with complex geometrical boundaries. It is true that there may be other advantages: computational efficiency is one which is often claimed. But even if these other advantages did not exist, the ability of the FEM to permit the analysis of geometrically complex physical situations would make it a potentially very powerful tool. In fluid mechanics, this potential has been revealed but is yet to be fully explored.

Huebner's book provides an excellent introduction for students and research workers wishing to enter this productive field.

G. DE VAHL DAVIS